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Coloring Face-Hypergraphs of Graphs on Surfaces

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The *face-hypergraph*, $\mathcal{H}(G)$, of a graph G embedded in a surface has vertex set $V(G)$, and every face of G corresponds to an edge of $\mathcal{H}(G)$ consisting of the vertices incident to the face. We study coloring parameters of these embedded hypergraphs. A hypergraph is k-colorable (k-choosable) if there is a coloring of its vertices from a set of k colors (from every assignment of lists of size k to its vertices) such that no edge is monochromatic. Thus a proper coloring of a face-hypergraph corresponds to a vertex coloring of the underlying graph such that no **face** is monochromatic. We show that hypergraphs can be extended to face-hypergraphs in a natural way and use tools from topological graph theory, the theory of hypergraphs, and design theory to obtain general bounds for the coloring and choosability problems. To show the sharpness of several bounds, we construct for every even n an n-vertex pseudo-triangulation of a surface such that every triple is a face exactly once. We provide supporting evidence for our conjecture that every plane face-hypergraph is 2-choosable and we pose several open questions, most notably: Can the vertices of a planar graph always be properly colored from lists of size 4, with the restriction on the lists that the colors come in pairs and a color is in a list if and only if its twin color is? An affirmative answer to this question would imply our conjecture, as well as the Four Color Theorem and several open problems.

Key Words: topological graph theory; weak coloring; hypergraph; faces; graphs on surfaces.

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1. INTRODUCTION

The first problem in coloring of graphs on surfaces, the Four Color Problem, was posed in 1852 by Francis Guthrie. He asked if one can always color a map with four colors so that neighboring countries receive distinct colors. Equivalently, can the vertices of every graph that embeds in the plane without crossings be colored with four colors so that no edge is monochromatic? This question was answered in the affirmative by Appel and Haken [4] in 1976, and the result is since known as the Four Color Theorem (see also [45]). Efforts to solve the Four Color Problem naturally led to the study of graphs that embed in surfaces other than the plane and to the general study of proper colorings, i.e., colorings of the vertices of a graph such that no edge is monochromatic.

A generalization of the notion of proper coloring that also posed an interesting problem for planar graphs was discovered independently by Vizing [51] and Erdős, Rubin, and Taylor [16]. Here the colors on the vertices have to be taken from lists placed at each vertex and the problem is to determine, for a given graph, the minimum list-size that always allows a proper coloring from a set of given lists. For general graphs, this problem is more challenging than the coloring problem. For planar graphs, Erdős, Rubin, and Taylor conjectured that lists of size 5 would always suffice. This conjecture was open for 15 years until Thomassen [48] provided a very elegant proof.

Both of these coloring notions can be extended to hypergraphs in a natural fashion. In a proper coloring of the vertices of a hypergraph no (hyper)edge is monochromatic (see Berge [6] for an introduction to hypergraph coloring). Problems for hypergraphs are more general and thus usually more difficult than those for graphs. From the historical perspective of graph coloring, however, it is natural to ask what coloring results can be obtained for "planar" hypergraphs, or more generally for hypergraphs that embed in a surface in some fashion.

In this paper, we consider a natural "embedded" hypergraph: the face-hypergraph $\mathcal{H}(G)$, which is derived from an embedding of a graph G in a surface. The vertex set of $\mathcal{H}(G)$ is that of G, and every face of G generates an edge in $\mathcal{H}(G)$ consisting of the vertices incident to the face. Thus, a proper coloring of the vertices of $\mathcal{H}(G)$ corresponds to a coloring of the vertices of G so that no face is monochromatic.

For planar hypergraphs, Burstein [11], Kostochka [29], and Penaud [39] independently proved that if all the edges are of size at least 3, then the vertices can be colored with 2 colors so that no edge is monochromatic. Since then however, this area has remained largely uninvestigated.

We begin our investigations by considering the list coloring problem for planar hypergraphs in Section 2. We generalize the problem to other surfaces in Section 3 and in Section 4 we provide some straight-forward graph theoretic bounds. A natural consequence of our investigations is the following intriguing conjecture about planar graphs.

Every plane face-hypergraph (with all edges of size at least 3) can be colored from lists of size 2.

We provide some evidence in favor of this conjecture in Section 4. We are also led to pose a list-coloring question for planar graphs, a positive answer to which would imply the conjecture and, among other things, the Four Color Theorem.

In Section 10, we prove that toroidal face-hypergraphs can always be colored from lists of size 3, and indeed the embedding of K_7 in the torus cannot be colored using only two colors. This makes the conjecture for the plane even more tantalizing. In Section 11, we use results from topological graph theory on "locally planar" graphs to show that face-hypergraphs whose underlying graphs have large edge-width can also be colored from lists of size 3.

In the other sections, we explore the problem from a hypergraph viewpoint. In Section 5 we formalize the hypergraph problem using the definitions of embedded graphs from Section 3. We then show that every connected hypergraph (without edges of size 1 or 2) can be extended to a face-hypergraph in a natural way while preserving its coloring and list coloring properties in Section 7. This allows us to use existing results on hypergraphs and triple systems to obtain bounds and complexity results for the coloring and list coloring problems.

Our results in some of these sections have a design-theoretic flavor. In particular, in Section 6 we give a necessary and sufficient condition for the existence of a pseudo-triangulation on n vertices in a surface so that every triple is a face exactly once. Our construction uses topological techniques of cutting and pasting and the graphs obtained show the sharpness of some of the bounds in Sections 4 and 5.

For completeness, we study the edge-coloring version of the problem in Section 12. We define the face-edge hypergraph, where the hyperedges correspond to the edge-sets bounding the faces of the graph embedded in the surface. We show that the face-edge hypergraph can be colored from lists of size 2 unless the graph is the dual of an odd cycle.

2. PLANE GRAPHS

We defer the formal statement of the general problem for hypergraphs and other definitions to later sections and consider the planar case first. For any undefined terms and concepts, we refer the reader to the introductory graph theory book of West [54].

Given an embedding of a loopless, undirected graph in the plane such that every face has at least three vertices, a *weak coloring* is a coloring of the vertices such that no **face** is monochromatic. Note that the usual notion of a *proper coloring* of a graph, which is a coloring of the vertices so that no **edge** is monochromatic, is also a weak coloring (unless the graph is edgeless).

Weak colorings of plane graphs have been studied in various contexts. Burstein [11], Kostochka [29], and Penaud [39] independently proved the following theorem. We present Penaud's elegant proof, an extension of which can be found in [31].

THEOREM 2.1. *Every plane graph is weakly 2-colorable.*

Proof. We may assume that the given graph G is a triangulation since adding edges, and therefore subdividing faces until they are triangles, does not make weak coloring easier. The dual graph G^* is a 2-connected, 3-regular graph, since G is a loopless triangulation. By Petersen's Theorem [54, p. 124], G^* has a 2-factor, i.e., a decomposition of its vertex set into cycles. We give $v \in V(G)$ color 0 if v is contained in the interior of an even number of cycles of the 2-factor and color 1 otherwise. To see that this yields a weak coloring, consider a face uvw in G. There is a pair of dual edges crossing this face, which belong to the 2-factor we obtained. These edges separate one of the vertices, say u, from the other two vertices in the face. Hence the parity of the number of cycles containing u is different from the parity of the number of cycles containing v and w. ▮

Suppose now that there are lists associated with each vertex of the plane graph G. If for every assignment of lists of size k there is a weak coloring using only the colors from the lists, then G is said to be *weakly k-choosable* or *weakly k-list-colorable*. The corresponding definition of k-choosability follows from replacing weak by proper above.

THEOREM 2.2. *Every plane graph is weakly 3-choosable.*

Proof. Again, we may assume that the given graph is a triangulation—if we choose a weak coloring of the triangulation from the lists, then we can remove the additional edges, and the coloring will still be a weak coloring of the original graph from the lists. To each of the lists of size 3 at every vertex, we add two dummy entries A and B so that the lists are now of size 5. By a theorem of Thomassen [48], plane graphs are properly 5-choosable, so there is a proper coloring of the vertices of G from the lists.

Consider the triangular faces of G under this coloring: if a face contains at most one dummy color on its vertices, then this face already has at least two vertices of distinct colors even before we replace the dummy color by some other color from the original list. Thus this face imposes no restriction on the color to be chosen at that vertex. So, we can concentrate on faces where both the dummy colors appear. Our goal is to replace A and B by distinct colors from the original lists, thus ensuring that these faces have at least two distinct colors. To do this, we consider the graph induced by the vertices of colors A and B. This is a bipartite plane graph and by a result of Alon and Tarsi [2] it is properly 3-choosable. Thus the vertices in this graph can be properly colored from the original lists of size 3. This modification of the previous coloring is a weak coloring of G. ▮

Clearly, the lists must have size at least 2, so it remains to be determined if every plane graph can indeed be weakly colored from lists of size 2 or not.

Conjecture 2.1. Every plane graph is weakly 2-choosable.

In Section 4 we provide some evidence for this conjecture. The following theorem verifies it in a special case.

THEOREM 2.3. *Every plane graph whose dual is Hamiltonian is weakly 2-choosable.*

Proof. Let G be given such that G^* is Hamiltonian. We may assume that G is a triangulation since we can triangulate G so that the Hamiltonicity of G^* is preserved. The Hamiltonian cycle C in G^* partitions $V(G)$ into two sets (one set of vertices V_1 inside C, and the other V_2 outside C). Both sets induce trees, since any cycle would enclose a face missed by C, and deleting the $2n-4$ edges of G crossed by C leaves $n-2$ edges for $G[V_1] \cup G[V_2]$. Since trees are properly 2-choosable [16] (this can be seen by successively removing leaves), we can properly color both trees from the given lists, thus ensuring that every edge in either tree has distinctly colored end-points. Now, any face of G must contain an edge in one of the two trees and hence cannot be monochromatic in the chosen coloring. ▮

This method can be extended to obtain weak 2-list colorings of plane graphs with duals having a 2-factor consisting of exactly two cycles. At present we are unable to extend this to triangulations whose duals have no 2-factors consisting of at most two cycles. The dual of every triangulation has a 2-factor; but for every k, the Tutte graph [54, p. 276] can be generalized to obtain a triangulation whose dual has no 2-factor consisting of at most k cycles (see also [46]).

Conjecture 2.1 is also related to the following question of Mohar and Škrekovski [37]:

QUESTION 2.1. *Let G be a plane graph such that every edge is in a triangle. Can G be colored from lists of size 2 such that no maximal clique is monochromatic?*

An affirmative answer to Question 2.1 would imply Conjecture 2.1 for triangulations that do not contain K_4 as a subgraph. If G contains K_4, then such a coloring need not be a weak coloring. For example, K_4 itself has a 2-coloring in this sense that is not a weak coloring.

3. SURFACES OF HIGHER GENUS

To study weak colorings of graphs embedded in surfaces other than the plane, we first present some definitions that delineate the topological requirements. For an introduction to topological graph theory, we refer the reader to the books of Gross and Tucker [21], and Mohar and Thomassen [36]. We consider only loopless, finite, undirected graphs embedded in surfaces where faces on 2 vertices are not allowed. In general the graphs we consider may have multiple edges, unless we specifically say that they are *simple*, i.e., without multiple edges.

Formally, for us a graph G is defined by (V, E, S, D), where V denotes the set of vertices, $V(G)$, E denotes the set of edges, $E(G)$, S is the surface in which G is embedded, and D is a drawing of G in S with no crossings. A *face* of G is an arc-wise connected component of $S-D$ and we require that all faces are homeomorphic to a disk, i.e., that D is an (open) 2-cell embedding. This implies, among other things, that G is connected.

Given an embedding of G in S, the *Euler genus* of $G(V, E, S, D)$ is defined as $\mathrm{eg(G)} = 2-|\mathrm{V(G)}|+|\mathrm{E(G)}|-|\mathscr{F}(\mathrm{G})|$, where $\mathscr{F}(G)$ denotes the set of faces of G. The surface S that G embeds in is *orientable* if it is homeomorphic to the sphere with $\mathrm{eg(G)}/2$ handles. It is *non-orientable* if it is homeomorphic to the sphere with $\mathrm{eg(G)}$ cross-caps.

The boundary of a face F consists of a *facial walk* of vertices and edges and its length $\ell(F)$ is the length of this walk. We also denote the set of vertices and edges in the boundary of F by $V(F)$ and $E(F)$, respectively. A face F has *size* $|E(F)|$ and *order* $|V(F)|$, which we denote by $e(F)$ and $n(F)$, respectively. Observe that for the embeddings we consider $\ell(F) \geqslant e(F) \geqslant n(F) \geqslant 3$ for every face, unless G is a tree. A graph in which every face has length exactly 3 is called a *pseudo-triangulation*. A graph is called a *triangulation* if it is simple and a pseudo-triangulation.

EXAMPLE 3.1. The embedding of K_4 in the sphere, which we denote by K_4^{sph} (see Fig. 1), is a triangulation, since every face has length 3. The order and size of every face is also 3. However the embedding of K_4 in the torus shown in Fig. 2, denoted by K_4^{tor}, has a face with length 8, size 6 and order 4.

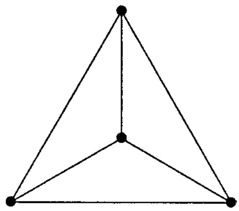

FIG. 1. K_4^{sph}.

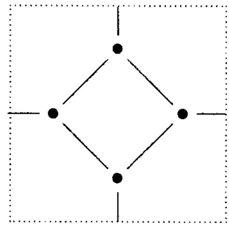

FIG. 2. K_4^{tor}.

Definition 3.1. A graph G is *weakly k-colorable* if we can color the vertices of G using at most k colors so that no face is monochromatic. (By this we mean that the vertex set of the face is not monochromatic.) The minimum integer k such that G is weakly k-colorable is the *weak chromatic number* of G, denoted by $\chi_w(G)$. If for every assignment of lists of size k to the vertices of G there is a weak coloring from the lists, then G is *weakly k-choosable*. The minimum integer k such that G is weakly k-choosable is the *weak list-chromatic number* of G, denoted by $\hat{\chi}_w(G)$.

If digon faces are allowed, then weak coloring can be identical to proper coloring. Hence for this paper we require every face to have order at least 3.

For coloring problems on embedded graphs, it is convenient to work with pseudo-triangulations. Triangulating a plane graph by adding edges is easy. Specifically, every simple plane graph can be extended to a simple triangulation on the same vertex set. This is not necessarily true for other surfaces as can be seen from embeddings of complete graphs that are not triangulations, such as K_4^{tor}. When multiple edges are allowed we can easily extend every embedded graph to a pseudo-triangulation.

Lemma 3.1. *Every embedded graph G can be extended to a pseudo-triangulation T of the same surface by adding edges such that $\chi_w(G) = \chi_w(T)$ and $\hat{\chi}_w(G) \leqslant \hat{\chi}_w(T)$.*

Proof. Suppose G is not a pseudo-triangulation. If we required only that $\chi_w(G) \leqslant \chi_w(T)$ then any extension would work, but we want to preserve the weak chromatic number. Consider a weak coloring of G with $\chi_w(G)$ colors. We will extend G to a pseudo-triangulation respecting the weak coloring. Let F be a face with $\ell(F) > 3$. We will add edges to F so that all new faces are 2-cells of order at least 3 and are not monochromatic under the given coloring. Repeating this process for each face of length more than 3 leads to the desired pseudo-triangulation.

If $n(F) > 3$ then pick three vertices $v, x, y \in V(F)$ not having the same color. Let v be a vertex whose color differs from the other two. Let z be

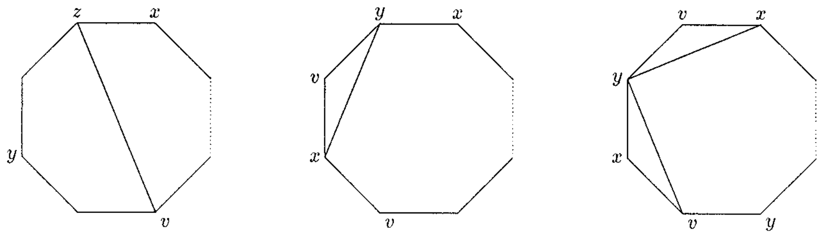

FIG. 3. Triangulating by adding edges.

any other vertex in $V(F)$. Consider the facial walk of F starting at v. We make a new edge from v to the vertex among x, y and z that appears second in this walk. This ensures that v is on both the new faces created and each face contains at least one of x or y (see Fig. 3).

So we may assume that $V(F) = \{u, v, w\}$. If in the facial walk we encounter a sequence of the form uvu, then we make an edge from w (anywhere in the walk) to v at this position to create two faces each containing u, v and w. In the remaining case, the facial walk must be of the form $uvwuvw\ldots$. Then, making an edge between any pair of distinct non-consecutive vertices works. ∎

4. GENERAL BOUNDS

We use the notions of proper coloring to obtain bounds on the weak chromatic and list-chromatic number of a graph. Recall that the *chromatic number* $\chi(G)$ of a graph G is the minimum k for which G has a proper coloring using k colors and the *list-chromatic number* $\hat{\chi}(G)$ is the minimum k such that G is properly k-choosable.

THEOREM 4.1. *If G is a pseudo-triangulation of an arbitrary surface, then $\chi_w(G) \leqslant \lceil \chi(G)/2 \rceil$ and $\hat{\chi}_w(G) \leqslant \lceil \hat{\chi}(G)/2 \rceil$.*

Proof. Suppose there is a list of size $\lceil \hat{\chi}(G)/2 \rceil$ at every vertex. Expand the lists by replacing every color c by two colors c_1 and c_2. We now have lists of size at least $\hat{\chi}(G)$, so there is a proper coloring of G from the lists. Erasing the subscripts yields a weak coloring of G from the original lists, since every triangular face consists of 3 pairwise adjacent vertices. The same proof also works for the first inequality, since we just assume that all lists are identical to $\{1, 2, \ldots, \lceil \chi(G)/2 \rceil\}$. ∎

Theorem 4.1 together with the Four Color Theorem gives an alternative proof for Theorem 2.1. With Thomassen's theorem [48] that planar graphs are 5-choosable, it also gives an alternative proof for Theorem 2.2.

Furthermore, we see that 4-choosable pseudo-triangulations of the plane are weakly 2-choosable. Examining the proof gives rise to the following "Paired-List Color" question:

QUESTION 4.2. *Can every plane graph be properly colored from arbitrary lists of size 4 with entries from* $\{a_1, a'_1, a_2, a'_2, ...\}$ *such that a list contains* a_i *if and only if it contains* a'_i?

It is worth noting that a positive answer to Question 4.2 would, besides answering Conjecture 2.1 and Question 2.1, also imply the Four Color Theorem. Furthermore, it would answer a question of Albertson, Grossman and Haas [1], who asked whether every planar graph can be colored from lists of size 2, such that there is a set of at least half the vertices inducing a properly colored graph. For graphs with Hamiltonian dual the proof of Theorem 2.3 can be extended to answer Question 4.2 in the affirmative: color the vertices in the first tree using only colors in $\{a_1, a_2, ...\}$ and for the second tree use only colors in $\{a'_1, a'_2, ...\}$. A plane graph that cannot be paired-list colored must be 5 list-chromatic, but various known examples of such graphs (e.g., [22, 34, 52]) in the plane are paired-list colorable [12]. For example, the dual of Voigt's example [52] is Hamiltonian.

Another immediate consequence of Theorem 4.1 is that $\hat{\chi}_w(G)$ and $\chi_w(G)$ are bounded in terms of the Euler genus of G. This was already observed by Burstein [11]; we give a slight strengthening of his result. Let $\bar{\delta}(G) = \max\{\delta(H) : H \subseteq G\}$.

THEOREM 4.2. *If G contains a subgraph H such that every facial walk in G contains the vertices of a cycle of H, then* $\chi_w(G) \leqslant \hat{\chi}_w(G) \leqslant \lfloor \bar{\delta}(H)/2 \rfloor + 1$.

Proof. We prove by induction on $n(H)$ that the vertices of such a subgraph H can be colored from lists of size $\lfloor \bar{\delta}(H)/2 \rfloor + 1$ such that no **cycle** has all vertices of the same color. The result then follows by coloring any remaining uncolored vertices arbitrarily, since every facial walk of G contains the vertices of a cycle of H.

If $n = 1$, then $H = K_1$ and one color suffices. Now suppose that $n(H) > 1$. Delete a vertex v of minimum degree from H. Since $\bar{\delta}(H - v) \leqslant \bar{\delta}(H)$, we can apply the induction hypothesis and color $H - v$ from the remaining lists so that no cycle is monochromatic. Next we note that at most $d(v)/2$ of the colors in the list of v create monochromatic cycles, since both neighbors of v on such a cycle would need to have the same color. Since the list at v has size at least $\lfloor \bar{\delta}(H)/2 \rfloor + 1$, there is a color available at v that can be used to complete the coloring of H. ∎

Suppose the graph G we consider has multiple edges. Since the above theorem requires that every facial walk of G contain the vertices of some cycle of a subgraph H, to form H we never need to take more than two copies of an edge. Two copies of an edge together form a cycle whose vertices are contained in any facial walk using the edge. Depending on the embedding of G (for example, if G is a pseudo-triangulation with multiple edges), H may even be taken to be a simple graph.

We present a construction showing the sharpness of Theorem 4.2 in Section 6. Theorem 4.2 gives yet another proof of Theorem 2.2, since every planar graph has a vertex of degree at most 5. It also implies that plane graphs of girth at least 4 (e.g., bipartite graphs) and outerplane graphs are weakly 2-choosable (since they have vertices of degree at most 3 and at most 2, respectively, and the families are hereditary). Furthermore, the following bound in terms of the Euler genus follows immediately (see also [30]):

COROLLARY 4.1. *If G is embedded in a surface other than the plane, then* $\hat{\chi}_w(G) \leqslant \lfloor (9+\sqrt{1+24\,\mathrm{eg}(G)}\,)/4 \rfloor$.

Proof. It is well know that a graph embedded in a surface of Euler genus eg has a vertex of degree less than or equal to $(5+\sqrt{1+24\,\mathrm{eg}}\,)/2$ for $\mathrm{eg} > 0$ (see [23]). ∎

General lower bounds for the weak chromatic number are more difficult. We use the following straightforward lemma to obtain a lower bound for triangulations.

LEMMA 4.1. *If G is a weakly 2-colorable triangulation of some surface, then* $|E(G)| \leqslant \frac{3}{8} |V(G)|^2$.

Proof. Consider a weak coloring of G. Let $|V(G)| = n$ and let t denote the size of one of the color classes. Since every edge is in exactly 2 faces and counting the edges crossing between color classes counts each face twice, we obtain $|\mathscr{F}(G)| \leqslant t(n-t) \leqslant n^2/4$. For a triangulation, $2\,|E(G)| = 3\,|\mathscr{F}(G)|$, so $|E(G)| \leqslant 3n^2/8$. ∎

Since $\binom{n}{2} \leqslant 3n^2/8$ only when $n \leqslant 4$, Lemma 4.1 immediately implies:

COROLLARY 4.2. *If K_n is embedded as a triangulation, then $\chi_w(K_n) \geqslant 3$ when $n \geqslant 5$.*

Together Corollaries 4.1 and 4.2 answer the weak choosability and coloring problems completely for the projective plane.

THEOREM 4.3. *If G embeds in the projective plane, then $\hat{\chi}_w(G) \leqslant 3$. This is optimal since a weak coloring of the embedding of K_6 in the projective plane as a triangulation, K_6^{pp}, requires* 3 *colors.*

Thus K_6^{pp} achieves the bound in Corollary 4.1. So far we have been unable to find graphs embedded in surfaces other than the projective plane for which the bound in Corollary 4.1 can be achieved. For graphs that embed in the Klein bottle, Corollary 4.1 gives an upper bound of 4 on their weak choosability number. However, these graphs are known to be 6-choosable. Hence, by Theorem 4.1, these graphs are actually 3-choosable. This is sharp—adding 3 edges to the embedding of K_6 in the Klein bottle gives a pseudo-triangulation that can be seen to require 3 colors.

In Section 10 we will see that Corollary 4.1 is not tight for the torus either, and in Section 6 we provide a construction that might suggest the correct order of magnitude. In the next section we present a bound in terms of the maximum degree Δ, which is at least asymptotically fairly tight.

5. FACE-HYPERGRAPHS

In this section, we formalize the weak coloring problem as a coloring problem for a special class of hypergraphs. A *hypergraph* $\mathcal{H} = (V, \mathcal{E})$ consists of a vertex-set $V = V(\mathcal{H})$ and a collection of edges $\mathcal{E} = \mathcal{E}(\mathcal{H})$, where each edge is a subset of V. A hypergraph is *r-uniform* if every edge is of size exactly r. The complete r-uniform hypergraph on n vertices has vertex set $V = \{1, 2, \ldots, n\}$ and edge-set $\mathcal{E} =$ all the possible r-sets from V. We denote it by $\mathcal{K}_n^r$.

Erdős and Hajnal [15] and Lovász [33] were apparently the first to consider vertex-colorings of what we now call hypergraphs. Somewhat later Berge [5] formalized the notion of the chromatic number of a hypergraph, under the term weak coloring.

DEFINITION 5.1. A *proper coloring* of $\mathcal{H}$ is a mapping of V into a set of colors such that no edge is monochromatic. A hypergraph that admits a coloring from a set of k colors is called *k-colorable*. The *chromatic number* $\chi(\mathcal{H})$ is the minimum k such that $\mathcal{H}$ is k-colorable. If for every assignment of lists of size k to the vertices of $\mathcal{H}$ we can color $\mathcal{H}$ from these lists, then $\mathcal{H}$ is *k-choosable* or *k-list-colorable*. The *list-chromatic number* $\hat{\chi}(\mathcal{H})$ is the minimum k such that $\mathcal{H}$ is k-choosable.

Clearly $\chi(\mathcal{H}) \leqslant \hat{\chi}(\mathcal{H})$. Furthermore, if we have repeated edges, then removing all but one copy of each edge from $\mathcal{E}(\mathcal{H})$ does not change the set of proper colorings, so that we usually think of $\mathcal{E}(\mathcal{H})$ as a set. We will investigate proper colorings of a special class of hypergraphs.

DEFINITION 5.2. The *face-hypergraph* $\mathcal{H}(G)$ of a graph G is the hypergraph with vertex-set $V(G)$ and edge-set $\{V(F) : F \in \mathscr{F}(G)\}$.

A proper coloring of $\mathcal{H}(G)$ clearly corresponds to a coloring of the vertices of G so that no face is monochromatic. Hence, by Definition 3.1 we see that $\chi_w(G) = \chi(\mathcal{H}(G))$ and $\hat{\chi}_w(G) = \hat{\chi}(\mathcal{H}(G))$. Thus in what follows, we can refer to a weak coloring (weak list-coloring) of G, or to a proper coloring (proper list-coloring) of $\mathcal{H}(G)$ interchangeably.

EXAMPLE 5.1. As seen in Figs. 1 and 2, the face-hypergraph $\mathcal{H}(K_4^{\text{sph}})$ has vertex set $\{1, 2, 3, 4\}$ and edge set $\{123, 124, 134, 234\}$, whereas $\mathcal{H}(K_4^{\text{tor}})$ has vertex set $\{1, 2, 3, 4\}$ and edge set $\{1234\}$. (In the second case, we drop the extra copy of edge 1234). Notice that $\mathcal{H}(K_4^{\text{tor}})$ is also the face-hypergraph for a 4-vertex tree or cycle in the plane, so the same face-hypergraph can be obtained from different graphs.

EXAMPLE 5.2. Two different embeddings of the same graph may have different weak coloring numbers. For example, Corollary 4.2 shows that triangulations of K_n for $n \geqslant 5$ are not weakly 2-colorable, whereas maximum genus embeddings of K_n have at most 2 faces [55] and therefore are weakly 2-colorable.

EXAMPLE 5.3. Not every hypergraph is a face-hypergraph. For example, the hypergraph with vertex set $\{1, 2, 3, 4, 5, 6\}$ and edge set $\{123, 456\}$ does not arise as the edge set of a face-hypergraph. If it does, then the underlying graph G must be connected (else it is not a 2-cell embedding). This requires some face to contain at least one vertex from $\{1, 2, 3\}$ and at least one vertex from $\{4, 5, 6\}$. Other small examples include $(\{1, 2, 3, 4, 5, 6\}, \{124, 135, 236\})$ and $(\{1, 2, 3, 4, 5\}, \{123, 124, 125, 345\})$.

Remark 5.1. A different notion of hypergraph embedding was investigated by Burstein [11], Jungerman, Stahl, and White [27], Walsh [53], Zykov [56], and others. The bipartite *incidence graph* $J(\mathcal{H})$ of a hypergraph $\mathcal{H}$ has bipartition $(V(\mathcal{H}), \mathscr{E}(\mathcal{H}))$, and the edge set of $J(\mathcal{H})$ consists of the pairs (v, E) such that $v \in E$. In this model, $\mathcal{H}$ is said to embed in a surface if and only if $J(\mathcal{H})$ embeds in the surface. Given an embedding of $J(\mathcal{H})$, one can obtain a graph G with vertex set $V(\mathcal{H})$ embedded in the surface such that every edge $E \in \mathscr{E}(\mathcal{H})$ is the edge set of a face in G that is a 2-cell bounded by a, cycle with vertex set E. Additional faces of G are 2-cells only if the embedding of $J(\mathcal{H})$ is a 2-cell embedding, which can be ensured as long as $J(\mathcal{H})$ is connected.

Remark 5.1 shows that every connected hypergraph can be embedded in a surface, but in general many additional faces are created that do not correspond to edges of $\mathcal{H}$, so that these embeddings do not have $\mathcal{H}$ as their face-hypergraph. In Section 7, we discuss a natural way of extending every hypergraph to a face-hypergraph while preserving its coloring properties.

We now present a general bound on the chromatic number of a face-hypergraph, that follows from a straight-forward application of the Lovász Local Lemma, see, e.g., [3].

THEOREM 5.1. *If $\mathcal{H}(G)$ is the face-hypergraph of an embedded graph G with maximum degree Δ, then $\hat{\chi}(\mathcal{H}(G)) \leqslant 1+\sqrt{e(3\Delta-2)}$, where $e=$ 2.7182....*

Proof. Assume that every vertex of $\mathcal{H}(G)$ has an associated list of size k. For each edge in $\mathcal{H}(G)$, pick three vertices. Since every edge in $\mathcal{H}(G)$ is a face of G, and we specified in Section 3 that all faces in our graphs have order at least three, we can do this. A vertex may be chosen for more than one face. Color these vertices by choosing a color for each vertex from its list, independently and uniformly.

For each edge $F \in \mathscr{E}(\mathcal{H})$, let A_F denote the event that all three vertices chosen from F get the same color under the random coloring. Then $\Pr(A_F) \leqslant 1/k^2$, and A_F is mutually independent of the set of all $A_{F'}$ such that the vertices chosen for F' are distinct from the vertices chosen for F. Since G has maximum degree Δ, a vertex can be chosen in at most Δ faces. Thus the number of faces that influence A_F is at most $3(\Delta-1)$. So, by the Local Lemma, there is a coloring of $\mathcal{H}(G)$ from the lists if $e(3\Delta-3+1)/k^2<1$. Thus, as long as $k>\sqrt{e(3\Delta-2)}$, there is a coloring of $\mathcal{H}(G)$ from the lists. ∎

6. COMPLETE FACE-HYPERGRAPHS

In this section we provide a construction which shows that the order of magnitude of Δ given in Theorem 5.1 is optimal. It also establishes the sharpness of Theorems 4.1 and 4.2.

We first observe that for 3-uniform hypergraphs $\mathcal{H}$ on n vertices $\chi(\mathcal{H}) \leqslant \lceil n/2 \rceil$ with equality if and only if either $\mathcal{H}$ is complete or $n(\mathcal{H})$ is even and $\mathcal{K}_n^3 \setminus \mathcal{H}$ is a set of intersecting edges that contains no $\mathcal{K}_4^3$. By Theorem 4.1, we know that $\chi_w(G) \leqslant \lceil \chi(G)/2 \rceil \leqslant \lceil n/2 \rceil$ for pseudo-triangulations G on n vertices. Thus to prove sharpness throughout it suffices to find a pseudo-triangulation whose face-hypergraph is $\mathcal{K}_n^3$. To keep the genus of the surface, and other parameters as small as possible, we might ask for an embedding of the n vertices that is a pseudo-triangulation such that every triple appears as a face **exactly once**. We call such an embedding complete.

DEFINITION 6.1. An embedding of a graph on n vertices is *complete* if

(1) it is a pseudo-triangulation,

(2) it has $\mathcal{K}_n^3$ as its face-hypergraph,

(3) it has exactly $\binom{n}{3}$ faces.

Only a complete graph (with multiple edges) can have a complete embedding. Since every edge is in exactly two faces and every pair of vertices must be in $n-2$ faces, we observe that the multiplicity of every edge must be $(n-2)/2$ and hence the embedding must be $\binom{n-1}{2}$-regular. Thus the Euler genus of such an embedding must be $(n-2)(n+3)(n-4)/12$.

The existence of a complete embedding shows that the order of magnitude of Δ in Theorem 5.1 is optimal, since it yields an upper bound of $\sqrt{3e\Delta}\,(1-o(1)) \leqslant n(2.02-o(1))$, which is within a factor of 4 of being optimal. Another consequence of the edge-multiplicity is that n must be even. To show that this necessary condition is also sufficient we define two operations. The topological details of these operations can be made rigorous by following the ideas of Hoffman and Richter [25].

Tetrahedron Pasting $T(u_1u_2, v_1v_2)$

Assume we have an embedding of a graph containing the four distinct vertices u_1, u_2, v_1 and v_2. Furthermore, suppose that that the edges u_1u_2 and v_1v_2 are present in the graph (Fig. 4a). We can now perform the *tetrahedron pasting* $T(u_1u_2, v_1v_2)$ as follows:

(1) Introduce double edges at u_1u_2 and at v_1v_2 to form digon faces (Fig. 4b).

(2) Cut the surface open along the digon faces just formed and reconnect the surface by inserting a tube (Figs. 4b and 4c).

(3) Introduce the edges u_2v_1, u_1v_1, u_2v_2, u_2v_1 on the tube as indicated by Fig. 4d. The picture indicates the boundary of the tube we just attached, i.e., the double-edges u_1u_2 and v_1v_2, with double lines and the tube is sliced up along the edge u_2v_1 so that this appears on top and bottom.

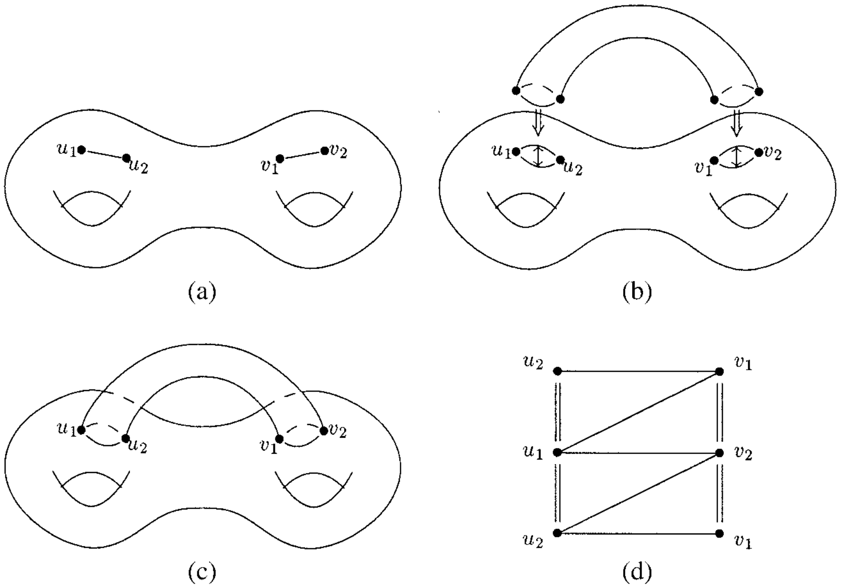

FIG. 4. Tetrahedron pasting $T(u_1u_2, v_1v_2)$.

The effect of $T(u_1u_2, v_1v_2)$ is as follows:

(1) If we start with a 2-cell embedding we obtain a 2-cell embedding.

(2) All old faces and edges are still present and we do not change the vertex set.

(3) We increase the genus by 1.

(4) We obtain exactly 1 new edge between every pair in $\{u_1, u_2, v_1, v_2\}$.

(5) We obtain exactly 1 new triangular face for every triple in $\{u_1, u_2, v_1, v_2\}$.

In other words the effect of $T(u_1u_2, v_1v_2)$ is that of adding all the edges and faces from the tetrahedron containing these 4 vertices.

The second operation adds all the edges and faces of an octahedron, where opposite vertices of the octahedron are denoted by x_1x_2, y_1y_2 and z_1z_2, respectively.

Octahedron Pasting $O(x_1x_2, y_1y_2, z_1z_2)$

Assume we have an embedding of a graph containing the six distinct vertices x_1, x_2, y_1, y_2, z_1 and z_2. This time we suppose that these 6 vertices can be covered by a matching of 3 edges that does **not** use any of the edges x_1x_2, y_1y_2 and z_1z_2. Without loss of generality, we can assume, as in Fig. 4a, that this matching is x_1y_2, y_1z_2 and z_1x_2. We can now perform the *octahedron pasting* $O(x_1x_2, y_1y_2, z_1z_2)$ as follows:

(1) Introduce a double edge at each of the matching edges x_1y_2, y_1z_2 and z_1x_2 to form digon faces (Fig. 5b).

(2) Cut the surface open along the three digon faces just formed and reconnect the surface by inserting a sphere with 3 holes as indicated in Figs. 5b and 5c.

(3) Introduce one new copy of every edge except x_1x_2, y_1y_2, z_1z_2 and the matching edges on the punctured sphere as indicated by Fig. 5d. Again the boundary of the punctured sphere is indicated by double lines and the tube is sliced up along the edges y_2z_2 and y_1z_1.

The effect of $O(x_1x_2, y_1y_2, z_1z_2)$ is as follows:

(1) If we start with a 2-cell embedding we obtain a 2-cell embedding.

(2) All old faces and edges are still present and we do not change the vertex set.

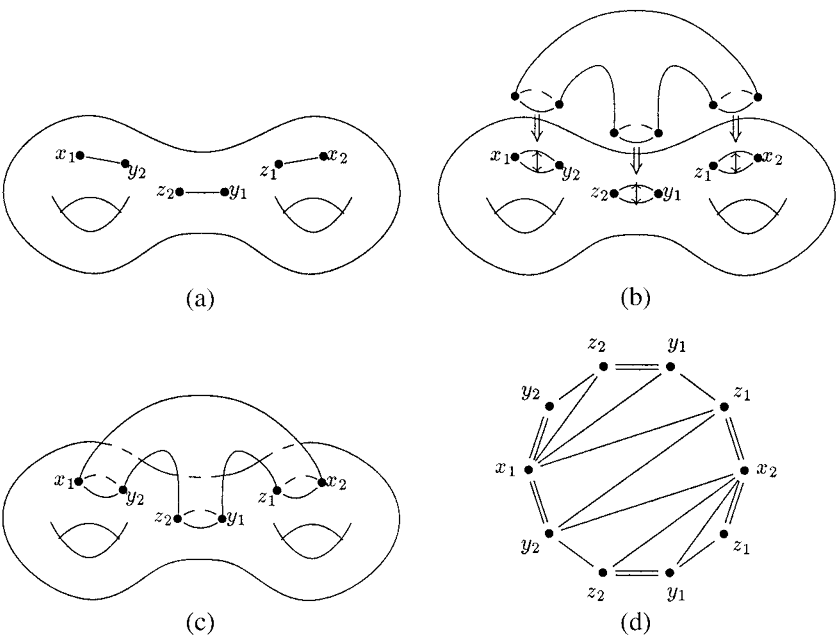

FIG. 5. Octahedron pasting $O(x_1x_2, y_1y_2, z_1z_2)$.

(3) We increase the genus by 2.

(4) We obtain exactly 1 new edge between every pair in $\{x_1, x_2, y_1, y_2, z_1, z_2\}$, except x_1x_2, y_1y_2 and z_1z_2.

(5) We obtain exactly 1 new triangular face for every triple of the form $x_i y_j z_k$.

Remark 6.1. The tetrahedron pasting can also be slightly modified as follows: Suppose v_1 and v_2 are not in the vertex set of the graph that we are augmenting, i.e., suppose we want to increase the vertex set by 2 vertices. In this case we can still create the digon involving u_1u_2 in (1) and attach the tube at this digon only in (2). In (3) we still use the same embedding, but in the end we simply identify the two copies of the edge v_1v_2. Thus the genus of the embedding did not increase and we created 2 new vertices, but all other consequences of the operation are unchanged. The octahedron pasting can be modified in a similar fashion.

THEOREM 6.1. *There is a complete embedding of a graph on n vertices if and only if n is even and at least 4.*

Proof. We already observed the necessity of n being even. For the sufficiency, let $n = 2\ell$ and let the vertices be v_i^j with $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant \ell$. Furthermore, let $V^j = v_1^j v_2^j$. We start with K_4^{sph} with vertex set $V^1 \cup V^2$. Next we perform $T(V^1, V^j)$ for every $3 \leqslant j \leqslant \ell$, so that we obtain an embedding of the vertex set in the plane. Now we perform $T(V^i, V^j)$ for every $2 \leqslant i < j \leqslant \ell$ and at this stage we already have a pseudo-triangulation of a complete graph on n vertices (with multiple edges) such that the faces are all triples containing two vertices from the same set V^j. In the last step we perform $O(V^i, V^j, V^k)$ for all $1 \leqslant i < j < k \leqslant \ell$ to establish the missing faces, those in which the three vertices are from different sets. ∎

Remark 6.2. Our construction gives a complete embedding in an orientable surface. To obtain an embedding in a non-orientable surface, we could replace one of the handles used in the tetrahedron pasting operation by a twisted handle [36, p. 82].

Besides showing that Theorems 4.1 and 5.1 are sharp, complete embeddings also give a sequence of graphs such that $\chi_w(G) = n/2 \approx \sqrt[3]{1.5\,\mathrm{eg}(G)}$, which may suggest the correct order of magnitude for Corollary 4.1. Furthermore, when applying Theorem 4.2 to a complete embedding on n vertices, we may assume that $H = K_n$ (i.e., the underlying *simple* clique), since every facial walk in the complete embedding contains a cycle of H. Thus complete embeddings achieve the bound in Theorem 4.2.

7. EMBEDDING HYPERGRAPHS INTO FACE-HYPERGRAPHS

We have already observed that not every hypergraph is a face-hypergraph, but we also mentioned in Remark 5.1 that every connected hypergraph can be extended to a face-hypergraph in a simple fashion. The following theorem makes this formal, and ensures that the embedding is nicely behaved.

THEOREM 7.1. *Let $\mathcal{H}=(V,\mathscr{E})$ be a connected hypergraph (without edges of size* 1 *or* 2*) with $n\geqslant 3$ vertices, e edges and total edge size $t=\sum_{E\in\mathscr{E}(\mathcal{H})}|E|$ (so that $\max\{e,n\}\leqslant t\leqslant en$). Then we can find a face-hypergraph $\mathcal{H}'=\mathcal{H}(G)$ in time $O(t)$, such that all of the following properties hold.*

(1) *$\mathcal{H}$ is an induced sub-hypergraph of $\mathcal{H}'$.*

(2) *$\chi_w(G)=\chi(\mathcal{H})$ and $\hat{\chi}_w(G)=\hat{\chi}(\mathcal{H})$.*

(3) *If $u,v\in V$, then the multiplicity of the edge uv in G is at most $|\{E\in\mathscr{E}:\{u,v\}\subset E\}|$.*

(4) *If $v\notin V$, then the degree of v in G is* 1.

(5) *For some $0\leqslant j\leqslant t/3$: $n(G)=n+j$, $e(G)\leqslant t+j$, $f(G)=e+j$ and* $\mathrm{eg}(G)\leqslant(\mathrm{t}-\mathrm{e}-\mathrm{n}-\mathrm{j}+2)/2$.

(6) *If we relax* (1) *by letting $\mathcal{H}$ be a sub-hypergraph of $\mathcal{H}'$ (not necessarily induced) and replace the equalities in* (2) *by $\chi_w(G)\geqslant\chi(\mathcal{H})$ and $\hat{\chi}_w(G)\geqslant\hat{\chi}(\mathcal{H})$, then in* (5) *we can let $j=0$.*

Proof. Form the incidence graph $J=J(\mathcal{H})$ as explained in Remark 5.1. Since $\mathcal{H}$ is connected, J is connected and J can be 2-cell embedded [36, Theorem 3.3.1], in some surface by specifying an embedding scheme [36, p. 92]. Since J is bipartite with bipartition $(V,\mathscr{E})$ the facial walk of every face alternates between vertices in V and $\mathscr{E}$. Since $\mathcal{H}$ has no edges of size 1, the vertex corresponding to each edge E has degree at least 2, so no such walk contains a sequence vEv. From J we now obtain a new graph J' by inserting, in every face, an edge between every pair of vertices $u,v\in V$ that are at distance 2 in the facial walk. Observe that we did not create loops or crossings and we obtain a 2-cell embedding of J'. Let $\mathscr{I}$ be the set of faces of J' which are not incident to vertices that come from edges in $\mathcal{H}$.

Next let G' be the subgraph of J' induced by V. Note that G' is 2-cell embedded. Faces in $\mathscr{I}$ remain unchanged. The triangular faces of J' incident to a single vertex $E\in\mathscr{E}$ together form a new face when we delete E. The boundary of this face is a cycle with vertex set $N_J(E)$, which equals E. Thus, we have embedded all the edges of $\mathcal{H}$ as vertex sets of faces in G'. However, the faces in $\mathscr{I}$ may still create problems since they could have order 2 and we also need to satisfy conditions (1) and (2).

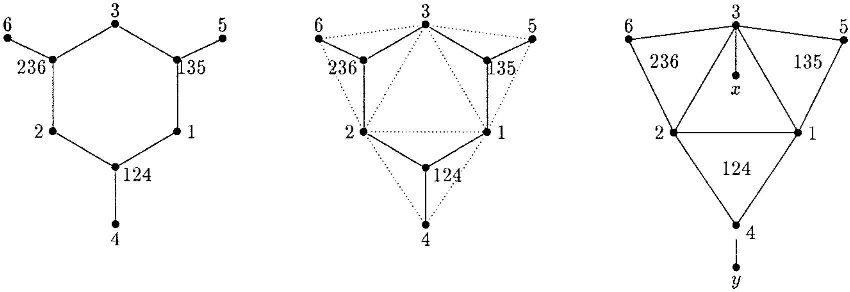

FIG. 6. Embedding a hypergraph in face-hypergraph.

Thus we modify G' slightly to obtain G as follows. In each face $F \in \mathscr{J}$ we introduce a new vertex v_F and make it adjacent to exactly one vertex in $V(F)$. The graph G we obtained is a 2-cell embedding and every face has order at least 3. Let $\mathcal{H}'$ be the face-hypergraph of G. It follows that (1) is fulfilled. Furthermore, (4) follows since every vertex not in V is adjacent to only one other vertex, and (3) follows from the construction of J'. (Observe that there need not be equality—if J has a face of length 4 of the form u, E, w, E', we only insert the edge uw once, even though u and w appear together in both E and E'.) Figure 6 illustrates the steps for the hypergraph with vertex set $\{1, 2, 3, 4, 5, 6\}$ and edge set $\{124, 135, 236\}$.

Next, let f_i be the number of faces of J of length i and j be the number of faces of J of length at least 6. It is easy to see that $j = |\mathscr{J}|$, so that G has $n+j$ vertices and $e+j$ faces, $t-f_4+j$ edges and the Euler genus now follows. Observe that $2t = 2e(J) = \sum_{i \geqslant 4} i f_i \geqslant 6j$, establishing (5).

For (2) observe that restricting a k-coloring of $\mathcal{H}'$ to V yields a k-coloring of $\mathcal{H}$. Conversely, given a k-coloring of $\mathcal{H}$, we can extend this to a k-coloring of $\mathcal{H}'$ by ensuring that each vertex in $V(\mathcal{H}')-V$ gets a color that is distinct from that of its only neighbor. The property of k-choosability follows similarly. Finally we observe that every step in the embedding process can be done in time $O(t)$. ∎

We can now apply Theorem 7.1 to show that for face-hypergraphs, as for graphs, there is no bound on how much $\hat{\chi}_w(\mathcal{H})$ can exceed $\chi(\mathcal{H})$ by, as n increases. It suffices to construct connected hypergraphs with this property. Then using Theorem 7.1, we can embed them and extend the result to face-hypergraphs. If we do not require our graphs to be simple, the construction given by Erdős, Rubin, and Taylor [16] can be easily extended to give a hypergraph with $\chi = 2$ but $\hat{\chi} = \ell$ with $(\ell+2)\binom{2\ell-1}{\ell}$ vertices [42]. Corollary 7.1 gives an alternative construction that can be extended to obtain a face-hypergraph whose underlying graph is simple.

COROLLARY 7.1. *There are graphs for which $\chi_w(G) < \hat{\chi}_w(G)$. Specifically, for every $2 \leqslant k \leqslant \ell$, there is a face-hypergraph $\mathcal{H}$ such that $\chi_w(\mathcal{H}) = k$ and $\hat{\chi}_w(\mathcal{H}) = \ell$.*

Proof. As remarked before, it suffices to exhibit connected hypergraphs with $\chi = k$ and $\hat{\chi} = \ell$. For each $\ell \geqslant 2$, we first construct a 3-uniform hypergraph $\mathcal{H}_1$ with $\chi(\mathcal{H}_1) = 2$ and $\hat{\chi}(\mathcal{H}_1) > \ell$. Let $V(\mathcal{H}_1)$ be the disjoint union of sets $U, A_1, \ldots, A_{\ell^\ell}$, where $U = \{u_1, u_2, \ldots, u_\ell\}$ and each A_i consists of $\ell + 1$ vertices. For each pair of vertices in some A_i and each u_j in U, we form an edge of $\mathcal{H}_1$ consisting these three vertices.

It is easy to see that $\mathcal{H}_1$ is 2-colorable—give all the vertices in U color 1 and the remaining vertices color 2. To show that it is not ℓ-choosable, we need to exhibit an assignment of lists of size ℓ to the vertices from which a proper coloring is not possible. Assign disjoint lists to the vertices in U. There are ℓ^ℓ possible lists that can be formed by taking one element from the list of each u_j. Assign the ith such list to each vertex of A_i.

We prove that no proper coloring can be chosen. For any choice of colors on U, there is some A_i, such that all the vertices in A_i have this choice of colors as their lists. Since each A_i has $\ell + 1$ vertices, some pair of vertices must get the same color. Hence there is a pair that has the same color as some u_j. This triple is a monochromatic edge in $\mathcal{H}_1$.

To modify $\mathcal{H}_1$ to a hypergraph with $\chi_w = k$ and $\hat{\chi}_w = \ell$ we first remove edges until $\hat{\chi}_w(\mathcal{H}_1) = \ell$. Then attach a copy of $\mathcal{K}^3_{2k}$ by identifying one of its vertices with some vertex of $\mathcal{H}_1$. Since $\chi(\mathcal{K}^3_{2k}) = k = \hat{\chi}(\mathcal{K}^3_{2k})$, this new hypergraph $\mathcal{H}$ has the desired values for $\hat{\chi}$ and χ and can then be extended to a suitable face-hypergraph $\mathcal{H}$ by Theorem 7.1. ∎

In extending the hypergraph in Corollary 7.1 to a face-hypergraph, we create an underlying graph that certainly has multiple edges. However, a slight modification of the construction yields a simple graph that is weakly 2-colorable, but not weakly ℓ-choosable. It suffices to ensure that in $\mathcal{H}_1$ no pair of vertices appears in more than one triple.

First increase U by taking $\ell 2^{2\ell}$ copies of every vertex u_j. Again, take disjoint lists at U but give each vertex in a group of $\ell 2^{2\ell}$ vertices the same list. This ensures that for every j at least $2^{2\ell}$ copies of u_j have the same color. Next we create $\binom{\ell 2^{2\ell}}{2^{2\ell}}^\ell$ copies of every A_i—one for every choice of $\ell 2^{2\ell}$ vertices from U, with $2^{2\ell}$ copies of every u_j. The vertices in A_i will only be in edges with the $\ell 2^{2\ell}$ vertices from U that A_i corresponds to. Finally, each A_i will consist of 2^ℓ vertices corresponding to the binary ℓ-tuples. If the first position where two vertices differ is position j, then create a triple using these two vertices and some copy of u_j. There are enough copies of every u_j so that we can assign a different copy to every pair. It is now straightforward to check that some triple must be monochromatic. Thus $\mathcal{H}_1$ yields a weakly 2-colorable simple graph that is not weakly ℓ-choosable.

8. SIMPLE GRAPHS AND TRIPLE SYSTEMS

Face-hypergraphs of simple graphs are closely related to triple systems. In general a triple system is merely a 3-uniform hypergraph, but we often impose a bound on the number of times a pair of vertices appears in a triple. We use this terminology in this section to stress the design theoretic flavor of the results given. The book of Colbourn and Rosa [14] is a comprehensive guide to triple systems. For a brief history of topological design theory, see [14, Sect. 0.4; 19].

DEFINITION 8.1. A *twofold triple system* is a triple system, such that every pair of vertices appears in **exactly two** triples. A *partial (twofold) triple system* is a triple system, such that every pair of vertices appears in **at most two** triples. A *Steiner triple system*, STS, is a triple system, such that every pair of vertices appears in **exactly one** triple. A *partial STS* is a triple system, such that every pair of vertices appears in **at most one** triple.

If G is a triangulation, then its face-hypergraph is a partial triple system: in fact, unless $G = C_3$, the pair uv appears in 2 triples of $\mathcal{H}(G)$ only if $uv \in E(G)$, and in no such triple otherwise. Already in 1891 Heffter [24] observed that a face-hypergraph of a triangulation of a complete graph is a twofold triple system and he gave rotation schemes to obtain embeddings for several complete graphs.

When seeking simple graphs with high weak-chromatic number, Lemma 4.1 and Theorem 4.1 suggest that triangulations of complete graphs are natural candidates to consider. We already know from Corollary 4.2 that triangulations of K_n have weak chromatic number at least 3, for $n \geqslant 5$. From Euler's formula for orientable surfaces, it follows that for an orientable embedding of K_n to be a triangulation, we must have $n \equiv 0$, 3, 4 or 7 mod 12. In 1968 Ringel and Youngs [43, 44] showed that these necessary conditions are also sufficient and their result triggered a flurry of activity in topological graph theory and design theory.

In Section 10 we will see that the unique embedding of K_7 in the torus is weakly 3-chromatic. Detailed case analysis yields that the known triangular embeddings of K_{19} [20, 21, 32] are weakly 4-chromatic. Interestingly, for all larger $n \equiv 7$ mod 12, the voltage graphs used in the standard construction [21, Figs. 5.5 and 5.7] can be used to obtain a weak 4-coloring of these embeddings of K_n. The face-hypergraphs of simple cliques are unlikely to achieve the bound in Theorem 4.1 within a constant factor. In fact they do not when the embedding is a triangulation.

To prove this, we use the following deep lemma to find a large enough independent set in a triple system on n vertices. Recall that a *cycle of length* k in a hypergraph is a list $(x_1, E_1, x_2, E_2, \ldots, x_k, E_k, x_1)$ alternating between vertices and edges of $\mathcal{H}$ such that the vertices are distinct, the edges are distinct, and $x_i, x_{i+1} \in E_i$ for all $1 \leqslant u \leqslant k-1$ and $x_k, x_1 \in E_k$.

LEMMA 8.1. (Komlós, Pintz, and Szemerédi [28]). *Let $\mathcal{H}$ be a triple system with k vertices and t triples that has no cycles of length at most 4. If for some $c_0 < c < k^{0.1}$ we have $t < c^2 k$, then $\mathcal{H}$ has an independent set of size at least $c^* \frac{k}{c} \sqrt{\log c}$, where c^* is an absolute constant.*

Phelps and Rödl [41] proved that a triple system where every pair is in at most 1 triple has an independent set of size at least $\sqrt{n \log n}$. The next theorem extends this to triple systems where every pair is in at most m triples.

THEOREM 8.1. *If $\mathcal{H}$ is a triple system on n vertices, with every pair of vertices in at most m triples, then $\mathcal{H}$ has an independent set consisting of $\Omega(\sqrt{(n/m)\log(n/m)})$ vertices.*

Proof. We closely follow the proof of Theorem 3.1 of [41], which corresponds to the case $m = 1$. We show that $\mathcal{H}$ has an induced sub-hypergraph with at least $\frac{1}{2}(n/m)^{5/9}$ vertices and at most $\frac{1}{5}(n/m)^{2/3}$ triples that has no cycle of length less than 5. We can then apply Lemma 8.1 with $k = \frac{1}{2}(n/m)^{5/9}$, $t = \frac{1}{5}(n/m)^{2/3}$ and $c = (1/\sqrt{2})(n/m)^{1/18}$.

Pick a random subset X of vertices by choosing $v \in X$ with probability $p = n^{-4/9} m^{-5/9}$. Thus $E(|X|) = pn = (n/m)^{5/9}$ and $\Pr(|X| < 0.9(n/m)^{5/9}) = o(1)$. Let $\mathcal{H}'$ denote the subhypergraph induced by X. For $2 \leqslant i$, let $C_i(X)$ denote the number of i-cycles in $\mathscr{E}(H')$ that do not contain a j-cycle for $2 \leqslant j < i$,

$$\begin{aligned} E(C_2(X) + C_3(X) + C_4(X)) &\leqslant \binom{n}{2}\binom{m}{2} p^4 + \binom{n}{3} m^3 p^6 + \binom{n}{4} (3m^4)\, p^8 \\ &= o((n/m)^{5/9}). \end{aligned}$$

Thus $\Pr(C_2(X) + C_3(X) + C_4(X) > 0.3(n/m)^{5/9}) = o(1)$. Similarly, since $E(|\mathscr{E}(H')|) \leqslant \frac{m}{3}\binom{n}{2} p^3 \leqslant \frac{1}{6}(n/m)^{2/3}$, we get $\Pr(|\mathscr{E}(H')| \geqslant 0.2(n/m)^{2/3}) \leqslant 5/6$. If we remove one vertex from every minimal cycle of length at most 4, then the probability that we find the desired sub-hypergraph is therefore at least $1/6 - o(1)$. ∎

THEOREM 8.2. *If G is a pseudo-triangulation on n vertices, with bounded maximum edge-multiplicity, then $\chi_w(G) = O(\sqrt{n/\log n})$. This bound is achieved by a simple graph.*

Proof. De Brandes, Phelps, and Rődl [10] have shown that there are k-chromatic Steiner Triple Systems with $O(k^2 \log k)$ vertices. If we embed such an STS using statement (6) of Theorem 7.1, then it follows from (3) and (4) that we obtain a simple graph, although not necessarily a triangulation. This shows that the given bound is essentially best possible.

The bound itself follows from the fact that pseudo-triangulations with bounded edge-multiplicities give rise to triple systems in which every pair of vertices is contained in a bounded number of triples. By Theorem 8.1, every such triple system on n vertices has an independent set of size $c\sqrt{n \log n}$. Iteratively taking maximum independent sets then yields the desired weak coloring, as it does in the proof of Theorem 4.1 in [41]. ∎

The construction in the proof of Theorem 8.2 is an embedding of K_n in which a weak coloring requires many colors. Since we saw in Example 5.2 that maximum (orientable) genus embeddings of K_n are weakly 2-colorable, this implies that the difference between the weak chromatic number of two different embeddings of the same graph can be arbitrarily large. It would also be interesting to see how large this difference can be if we fix the surface.

Conjecture 8.1. For each k, there is a graph that has two different embeddings in the same surface whose weak chromatic numbers differ by at least k.

It seems likely that Conjecture 8.1 holds even if the graph is embedded in a surface of minimum genus. This is supported by the intriguing result of Bonnington *et al.* [9] that complete graphs can have many different orientable embeddings of minimum genus. They show that for $n \equiv 7$ or 19 (mod 36) there are at least $2^{n^2/54 - O(n)}$ non-isomorphic triangular embeddings of K_n.

9. ALGORITHMIC COMPLEXITY

In this section we show that in general determining the weak coloring number of a face-hypergraph is NP-complete.

THEOREM 9.1. *For fixed k at least 2, deciding whether a given simple graph is weakly k-colorable is NP-complete.*

Proof. The problem is certainly in NP. Colbourn *et al.* [13] have shown that, for every fixed $k \geqslant 3$, it is NP-hard to decide if a partial STS $\mathcal{H}$ is k-colorable. Since Theorem 7.1 gives a polynomial time reduction of testing $\chi(\mathcal{H}) \leqslant k$ to testing $\chi_w(G) \leqslant k$ for a simple graph G, this proves the result for $k \geqslant 3$.

For $k=2$, Phelps and Rödl [40, Corollary 6.2] observe that 2-coloring a general 3-uniform hypergraph can be reduced to 2-coloring a partial STS in polynomial time. Thus it suffices to show that 2-coloring 3-uniform hypergraphs is NP-hard, which follows for example, from NOT-ALL-EQUAL 3-SAT: let $V(\mathcal{H})$ be the set of literals plus 3 extra vertices a, b, c, and let $\mathscr{E}(\mathcal{H})$ consist of all clauses (as triples of vertices), plus abc and for every variable x the triples $x\bar{x}a$, $x\bar{x}b$ and $x\bar{x}c$. Now a 2-coloring of $\mathcal{H}$ corresponds in a natural fashion to an assignment of truth values such that every clause has at least one true and one false literal, since the extra triples guarantee that x and $\bar{x}$ get opposite colors. ∎

The graphs that show that the general problem is NP-complete have increasingly large Euler genus. This suggests a more precise question about the boundary between P and NP.

QUESTION 9.3. *For which values of* eg *and k is it decidable in polynomial time if a given embedded graph of Euler genus* eg *is weakly k-colorable?*

For $\mathrm{eg}=0$, and for $\mathrm{eg}=1$ and $k=3$, Theorems 2.1 and 4.3, respectively, answer the question. For $\mathrm{eg}=2$ we will show in Theorem 10.2 that 3 colors always suffice. For both the torus and the projective plane, however, the question for $k=2$ remains open. For $k \geqslant 5$, Thomassen [50] (also [36, Corollary 8.4.2]) has shown that it is decidable in polynomial time if a graph on a fixed surface is properly k-colorable, whereas proper 3-colorability is NP-complete for planar graphs [18]. Unfortunately the weak coloring problem differs from the proper coloring problem in that proper subgraphs may have higher weak chromatic number (to see this insert a vertex into every face of a graph with high weak chromatic number and make the graph connected—the resulting graph is weakly 2-colorable). This makes an approach as used in [50] difficult.

10. TOROIDAL GRAPHS

Corollary 4.1 shows that all toroidal graphs are weakly 4-choosable. A good candidate for a graph with high weak choosability is the embedding of K_7 in the torus shown in Fig. 7, K_7^{tor}.

THEOREM 10.1. $\chi_w(K_7^{\mathrm{tor}}) = 3 = \hat{\chi}_w(K_7^{\mathrm{tor}})$.

Proof. It follows from Corollary 4.2 that K_7^{tor} is not weakly 2-colorable. A weak 3-coloring is given by the color classes $\{1\}$, $\{234\}$, $\{567\}$. To see

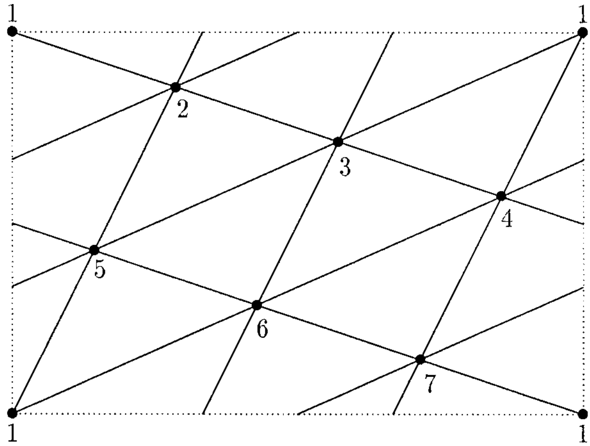

FIG. 7. K_7^{tor}.

that K_7^{tor} is weakly 3-choosable, observe that every face contains two vertices of the form $i, i+1 \pmod 7$. So if we start by giving vertex 1 any color, then successively color vertices 2 through 6 by giving them a color different from their predecessor and finally color vertex 7 with a color different from vertices 1 and 6, then we obtain a weak coloring of K_7^{tor} from the lists. ∎

This raises the question whether toroidal graphs are in general weakly 3-choosable. To show that this is indeed the case we need the following lemma, which is a strengthening of Theorem 2.2.

LEMMA 10.1. *Let G be a plane graph such that every face has at least* 3 *vertices. Then G can be weakly colored from lists of size 3, even if up to* 3 *vertices are precolored (with at most* 2 *getting the same color).*

Proof. We prove the theorem by induction on $n(G) \geqslant 3$. By Lemma 3.1 we may assume that G is a pseudo-triangulation. Thus, for $n(G) = 3$, G must be a 3-cycle and the result holds.

Assume $n(G) > 3$. Suppose that G contains a vertex v of degree at most 5 that is not precolored. Delete v; if the resulting graph has any faces of size 2, remove one copy of the edge from each such face. Call the resulting graph G'; it satisfies the hypothesis and can be weakly colored from the lists. This coloring can then be extended to a weak coloring of G by picking a color for v that appears on at most one of its neighbors. (Since $d(v) \leqslant 5$, there is such a color). The result is a weak coloring of G because in each face containing v the color on v appears on at most one of its neighbors.

Hence we may assume that all the vertices of degree at most 5 are precolored. By hypothesis, there are at most three such vertices; we reduce the claim to the case where there are exactly 3 precolored vertices and they are all of degree 2. This follows from Euler's formula. Since G is a pseudo-triangulation, $\sum_{v \in V(G)} (6-d(v)) = 12$. If S is the set of all vertices that are of degree at most 5, $\sum_{v \in S} (6-d(v)) \geqslant 12$. However, $6-d(v) \leqslant 4$ and $|S| \leqslant 3$, so $\sum_{v \in S} (6-d(v)) \leqslant 12$. Thus equality holds, and the claim follows.

One of the precolored vertices v must be adjacent to a vertex w that is not precolored, since otherwise $G = C_3$. Color w with a color different from v and different from at least one of the other precolored vertices. Again, consider the graph G' obtained by removing v and removing one copy of an edge from any faces of size 2 that are created when v is removed. Now G' satisfies the hypothesis: it has only three precolored vertices and we can weakly color it from the lists. Replacing v we obtain a weak coloring of G since v can be involved in only one face and that face contains w. ∎

THEOREM 10.2. *If G is a toroidal graph, then $\hat{\chi}_w(G) \leqslant 3$.*

Proof. It suffices to prove the result for pseudo-triangulations. If $\hat{\chi}(G) \leqslant 6$, then the result follows from Theorem 4.1; hence we may assume that $\hat{\chi}(G) = 7$. In this case it follows from a theorem of Böhme, Mohar and Stiebitz [7] that G contains K_7 as a subgraph. Since K_7 can only be embedded as K_7^{tor} in the torus it can, by Theorem 10.1, be weakly colored from its lists. Every face of K_7^{tor} is now a 2-cell bounded by a triangle with at least 2 distinct colors. Hence the vertices in the interior of every face can be weakly colored from the lists by Lemma 10.1. ∎

Remark 10.1. The fact that K_7 embeds uniquely in the torus follows for example from the general result of Negami [38] that every 6-connected graph in the torus is uniquely embeddable. A different proof is that it is easily seen that K_7 must form a triangulation of the torus and thus give rise to a twofold triple system. However, of the 4 non-isomorphic twofold triple systems on 7 vertices [14, Table 5.3] only one comes from an embedding in the torus.

11. LOCALLY PLANAR FACE-HYPERGRAPHS

We call a graph *locally planar* if, for every vertex v, the subgraph induced by the vertices close to v is a plane graph. Equivalently the graph has no short non-contractible cycles. The length of a shortest non-contractible cycle in G is the *edge-width* of G, $\mathrm{ew}(G)$. Graphs with large edge-width have received considerable attention in recent years. The following result is due to Thomassen.

THEOREM 11.1 (Thomassen [47, Theorem 5.7]). *Let G be embedded in an orientable surface S with Euler genus* eg. *If* $\mathrm{ew}(G) \geqslant 2^{7\mathrm{eg}+6}$, *then* $\chi(G) \leqslant 5$.

With Theorem 4.1 this implies that triangulations with large edge-width are weakly 3-colorable. This result can be improved by using a more recent result of Thomassen that holds for both orientable and non-orientable surfaces.

THEOREM 11.2 (Thomassen [50, Theorem 4.4]). *Every graph G with $\hat{\chi}(G) \geqslant 7$ that is embedded in a surface of Euler genus* eg *contains a subgraph on at most* 75eg *vertices with the same choice-number.*

COROLLARY 11.1. *Let G be a triangulation of a surface S with Euler genus* eg. *If* $\mathrm{ew}(G) > 75\mathrm{eg}$, *then* $\hat{\chi}_w(G) \leqslant 3$.

Proof. By Theorem 4.1 it suffices to show that $\hat{\chi}(G) \leqslant 6$. If this is not the case, then by Theorem 11.2, G has a subgraph H of order at most 75eg such that $\hat{\chi}(H) \geqslant 7$. Since $\mathrm{ew}(G) > 75\mathrm{eg}$, H has no noncontractible cycles, and therefore H is planar. This contradicts the 5-choosability of planar graphs [48]. ∎

Since graphs with edge-width more than 75eg are weakly 3-colorable, it is natural to ask if sufficiently high edge-width (in terms of the genus) can guarantee weak-2-colorability.

QUESTION 11.4. *Is there a function f, such that* $\mathrm{ew}(G) \geqslant f(\mathrm{eg}(G))$ *implies that* $\chi_w(G) \leqslant 2$?

The answer is yes for orientable Eulerian triangulations. A recent result of Hutchinson, Richter, and Seymour [26] states that for every orientable surface, there is a number c, depending on the surface, so that every Eulerian triangulation of the surface with edge-width at least c is 4-colorable. The situation is different for Eulerian triangulations of non-orientable surfaces [35].

A cycle is *non-bounding* (also called *non-separating*) if removing it does not disconnect the surface, and we denote the length of a shortest non-bounding cycle of G by $\mathrm{nbd}(G)$. Since every non-bounding curve is non-contractible we have $\mathrm{ew}(G) \leqslant \mathrm{nbd}(G)$, but equality need not hold. Fisk and Mohar [17] proved that graphs without very short non-bounding cycles are 6-colorable, so we obtain:

COROLLARY 11.2. *There is a constant $c > 0$, which is independent of* eg, *such that every triangulation G of S_{eg} with* $\mathrm{nbd}(G) > c\log(\mathrm{eg}+1)$ *is weakly 3-colorable.*

In fact, the proof in [17] can be extended to show 6-choosability of such graphs, so these graphs are even 3-choosable. The growth rate $c\log(\gamma+1)$ obtained by Fisk and Mohar is optimal for proper colorings. We observe that for both corollaries we need G to be a pseudo-triangulation—triangulating a graph may reduce the length of a shortest non-contractible cycle. If however we require that G has large representativity instead, then we may drop the requirement that G is a pseudo-triangulation. The *representativity* of a graph on a surface is the minimum number of times that any non-contractible curve meets the drawing. Hence triangulating a graph only increases its representativity, and the representativity of a pseudo-triangulation is exactly its edge-width. In general, the representativity of a graph is no larger than its edge-width.

12. EDGE-COLORING WITH FACE CONSTRAINTS

In the last section we investigate the problem of coloring the **edges** of an embedded graph such that every face is incident to at least two edges of different color. We have seen that the vertex version gives rise to a number of challenging questions. For the edge-version the situation is much simpler: we show that we can color the edges of an embedded graph from lists of size 2, even if we allow loops, faces of length 2 and faces that are not 2-cells, unless its dual is an odd cycle or contains a face of size 1.

DEFINITION 12.1. The *face-edge-hypergraph* $\mathcal{H}'(G)$ is the hypergraph with vertex set $E(G)$ and edge-set $\{E(F) : F \in \mathscr{F}(G)\}$. We define $\chi'_w(G) = \chi(\mathcal{H}'(G))$ and $\hat{\chi}'_w(G) = \hat{\chi}(\mathcal{H}'(G))$.

The problem of list coloring the face-edge hypergraph can be phrased using the dual graph in a way that eliminates the topological considerations. We want to color the edges of the dual from the lists so that each dual vertex is incident to at least two edges of different color. If the dual graph is a loop or an odd cycle with identical lists of size 2, then this is clearly not possible. We call such a cycle a *bad cycle*.

Note that since the original graph has no faces of size 1, its dual has no vertices of degree 1. Thus our desired theorem follows immediately from the following result, whose proof is similar to Lemma 6.1.1 in Bondy and Murty [8]. We include the proof for completeness.

THEOREM 12.1. *Let G be a connected multi-graph (not necessarily loopless) with lists of size 2 assigned to each edge. Then, there is a coloring of the edges from their lists such that each vertex incident to at least two edges* *is incident to edges of at least two distinct colors if and only if G is not a bad cycle.*

Proof. We already saw the necessity of G not being a bad cycle. If G is a cycle with not all lists identical, then some pair of incident edges e_1, e_n get different lists and we can assign a color to e_1 that is not in e_n's list. Now color the edges $e_2, e_3, \ldots, e_n$ in this order by choosing a color for e_i that is different from the color chosen on e_{i-1}. If G is a cycle with identical lists, then the problem reduces to properly 2 edge-coloring a cycle.

Next suppose that G is an Eulerian graph, with a vertex v_0 of degree at least 4. Let $v_0 e_1 v_1 \cdots e_m v_0$ be an Eulerian cycle starting at v_0 and color e_1 from its list. Again, coloring every remaining edge with a color from its list that differs from the color on the previous edge yields the desired coloring.

Finally assume that G is not Eulerian. Construct a new graph G' by adding a new vertex v_0. We make v_0 adjacent to every odd-degree vertex in G, and assign arbitrary lists of size 2 to the new edges. Now G' is Eulerian and the same coloring as in the previous paragraph works. ∎

Thus we obtain

THEOREM 12.2. *Let G be a graph embedded in a surface (not necessarily 2-cell) such that every face has size at least* 2. $\hat{\chi}'_w(G) = \chi'_w(G) = 2$, *unless* G^* *is an odd cycle, in which case both parameters take the value* 3.

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